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Soft gluons at large angles in hadron collisions

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Abstract: A general discussion is presented of the single logarithmic soft factor that appears in two scale QCD observables in processes involving four partons. We treat it as the "fifth form factor", accompanying the four collinear singular Sudakov form factors attached to colliding and outgoing hard partons. The fifth form factor is expressed in terms of the Casimir operators (squared colour charges) of irreducible representations in the crossing t- and u-channels. As an application we revisit the problem of large angle radiation in $gg \to gg$ and give a relatively simple solution and interpretation of the results. We found an unexpected symmetry of the soft anomalous dimension under exchange of internal and external variables of the problem whose existence calls for explanation.

Keywords: Hadronic Colliders, QCD, Jets.

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Contents

1. Introduction

QCD observables that are characterised by two large scales $Q_0 \gg \Lambda_{\mathrm{QCD}}$ and $Q \gg Q_0$ are sensitive to multiple gluon radiation and possess double logarithmic (DL) quark and gluon form factors depending on the ratio of the scales, $\alpha_{\mathrm{s}} \ln^2(Q/Q_0)$. Here Q is the overall hardness scale of the process which is determined by the underlying parton-parton interaction, and the smaller scale Q_0 is introduced by measuring an observable V in specific kinematics.

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Hard interaction induces associated emission of relatively soft and/or quasi-collinear gluons. We discuss observables V that measure various characteristics of the *secondary parton ensemble* in inclusive manner. Examples are thrust ($V = 1-T$) and broadening ($V = B$) event observables in e^+e^- annihilation and DIS, accumulated out-of-event-plane momentum in three-jet e^+e^- events as well as in hard hadron collisions producing jets.

Such observables vanish at the Born level (pure underlying parton event), $V = 0$, and may reach $V = \mathcal{O}(1)$ in the presence of secondaries that are energetic and non-collinear to primary parton directions and look as additional jets. On average, $\langle V\rangle = \mathcal{O}(\alpha_s)$. Restricting the observable even further, $V \ll \langle V\rangle$, introduces the second scale Q_0 setting the maximal allowed transverse momentum of *real* secondary partons. At the same time, transverse momenta of *virtual* gluons are not bounded. Break-up of the real–virtual cancellation gives rise then to the DL form factor suppression of near-to-Born parton configurations.

DL enhanced corrections originate from emission of soft collinear gluons. They have simple transparent physical origin, which helps to analyse and resum them in all orders into exponential Sudakov form factors attached to each of primary hard partons.

Subleading *single logarithmic* (SL) contributions originate from various sources. First of all, the DL effects must be treated with care in order to precisely define the arguments of the DL functions. Moreover, in certain cases SL corrections emerge due to "recoil" that secondary radiation produces upon the primary partons, which affects determination of the thrust axis or of the event plane. These SL effects are due to precision treatment of kinematics (definition of the observable, global momentum conservation, etc.) and are basically of DL nature. Moreover, there are direct SL contributions that are suppressed at the matrix element level and originate either from collinear hard parton splittings ($z \sim 1$) or from radiation of soft gluons at large angles. The former is an intrinsic part of jet evolution and is easy to account for. The latter — inter-jet radiation — poses, in principle, more problems.

In particular, n-parton ensembles consisting of energy ordered gluons radiated at large angles contribute, for example, to particle energy flow $Q_0 = E$ in a given inter-jet direction at the SL level as $\mathcal{O}(\alpha_s^n \ln^n(Q/Q_0))$ as was found in [1]. Such "hedgehog" multi-gluon configurations are difficult to analyse. The all-order results for such (so called "non-global") observables were obtained only in the large-N limit [1, 2].

Global observables that acquire contributions from the full available phase space (rather than from a restricted phase space "window" as the non-global ones do) are free from this trouble: only the hardest among the gluons contributes while the softer ones don't affect essentially the observable and their contributions cancel against corresponding virtual terms. As a result, contribution of large angle soft gluon radiation reduces to virtual corrections due to multiple gluons with $k_t > Q_0$ attached to primary hard partons. They can be treated iteratively and fully exponentiated, together with DL terms.

Distributions in various global observables have been resummed in all orders, with SL accuracy, in the case of underlying QCD processes involving two or three partons that one finds in e^+e^-, DIS and in hadron-hadron collisions with a hard electro-weak object in the final state (large-p_T photon, Drell-Yan pair, Z^0 boson, etc.).

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Carrying out this extensive programme was simplified by the fact that soft gluon radiation in two- and three-patron systems is essentially *colour-trivial*. Indeed, let T_i be the colour generator that enters the amplitude of soft gluon radiation off a hard parton i. Then, due to the colour current conservation ($T_1 + T_2 = 0$ for two and $T_1 + T_2 + T_3 = 0$ for three participating hard partons), the products $T_i T_j$ that enter the soft gluon radiation probability off the underlying primary partons are *proportional to the identity* in the colour space. As a result, the answer could be expressed as the product of Sudakov *form factors* corresponding to each of the primary partons. Each form factor $F_i(Q_i, Q_0)$ is collinear singular and given by exponent of the probability of single gluon emission proportional to the "colour charge" of the hard parton i. Importantly, this answer takes full care of both DL and SL effects provided one introduces into the form factors properly defined hardness scales Q_i that depend on the event geometry.

The case of four participating partons is the first one when colour triviality no longer holds [3]. Here two colliding partons can be found in various colour states. Radiation of a gluon changes the colour state of the parton pair and this affects radiation of the next (softer) gluon. Successive large angle gluon emissions become interdependent. As a result the product of independent Sudakov form factors misses essential SL corrections. The programme of resumming soft SL effects due to large angle gluon emission in hadron-hadron collisons was addressed in a series of papers [3 − 6]. It gives an additional form factor which is not diagonal in colour indices.

This does not constitute a serious technical problem for parton collisions involving quarks (quark scattering and annihilation into gluons, QCD Compton) since a $q\bar{q}$ pair can be only in two colour states, **8** and **1**, translating into **6** and $\mathbf{\bar{3}}$ for a qq system. Gluon-gluon scattering is more involved: here one finds as many as *five* irreducible colour representations: $\mathbf{8_a}$, $\mathbf{10} + \overline{\mathbf{10}}$, **1**, $\mathbf{8_s}$, **27** (*six* in the general $SU(N)$ case). The problem of *diagonalisation* of the system of mixing colour channels in gg scattering was formulated by George Sterman and collaborators in [4].

In what follows we present a transparent physical interpretation of large angle radiation effects which can be expressed in terms of the "*fifth form factor*" that depends on charge exchange in the cross (t- and u-) channels of the scattering process. We have summarized part of the results in [7]. With account of logarithmically enhanced, DL and SL, virtual corrections the matrix element M_0 becomes

$$M_0 \;\to\; \prod_{i=1}^{4} F_i(Q_i, Q_0) \cdot F_X(\tau_0) \cdot M_0 \;\equiv\; \prod_{i=1}^{4} F_i(Q_i, Q_0) \cdot M(\tau_0)\,. \tag{1.1}$$

The fifth form factor F_X (where "X" stands for "cross-channel") is collinear finite and therefore a SL function depending on the logarithmic variable

$$\tau_0 = \int_{Q_0}^{Q} \frac{dk_t}{k_t} \frac{\alpha_s(k_t)}{\pi}\,. \tag{1.2}$$

It also depends on kinematical variables s, t, u of the hard process.

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The fifth form factor exists in the QED context as well, but the specificity of QCD scattering is that here it has essentially non-abelian structure since its exponent contains non-commuting t- and u-channel (squared) colour charges. It reduces, however, to the Abelian construction in special case of small angle scattering, $|t| \ll s$ or $|u| \ll s$, where it describes reggeization of corresponding cross channel exchange states (in particular, gluon and quark reggeization).

We also remark upon the existence of a strange symmetry between the external (scattering angle) and internal variables (rank of the gauge group). Elucidating the nature of this unexpected symmetry calls for an effort on the part of the "theoretical-theory" community.

It is clear that the programme of resumming soft SL effects due to large angle gluon emission requires special care to be taken of collinear enhanced DL contributions that have to be treated with subleading SL accuracy. This involves precise definition of the arguments Q_i of the DL form factors and of the parton densities, an accurate account of running coupling effects, employing Mellin/Fourier transformation to carefully factorise multiple emissions, etc. Carrying out this programme results in the expression for the (integrated) distribution $\Sigma(Q_0, Q)$ that has the following general form:

$$\Sigma(Q_0, Q) \;=\; \Sigma^{\rm coll}(Q_0, Q) \cdot \mathcal{S}_X(\tau_0)\,. \tag{1.3}$$

Here the "collinear factor" $\Sigma^{\rm coll}$ originates from the four Sudakov form factors in (1.1) and embodies the parton densities, while the SL soft factor $\mathcal{S}_X$ is due to the cross-channel "fifth form factor" F_X,

$$\mathcal{S}_X(\tau_0) \;=\; \frac{{\rm Tr}(M^\dagger(\tau_0)\, M(\tau_0))}{{\rm Tr}(M_0^\dagger\, M_0)}\,, \qquad M(\tau_0) = F_X(\tau_0) \cdot M_0\,. \tag{1.4}$$

The paper is organised as follows. In the next section 2 we give the general treatment of soft gluon radiation accompanying hard scattering of arbitrary colour objects. We elucidate the origin of the soft SL factor ("fifth form factor") as due to large angle radiation which is governed by colour being exchanged in crossing channels. We describe the general procedure for calculating the corresponding soft anomalous dimension in terms of Casimir operators of t- and u-channel colour states. We also demonstrate cancellation of the divergent piece of non-abelian Coulomb phase.

Section 3 is devoted to gluon-gluon scattering. Here we give an ergonomic solution for eigenvalues and eigenvectors of the anomalous dimension matrix and present the answer for the soft factor in various special cases.

The main ingredients of the paper are collected in Conclusions where in particular we draw reader's attention to unexpected symmetry of the problem.

2. Soft gluons and the fifth form factor

Consider a hard process involving N_p primary partons. Let T_i^b be the colour generator that enters the amplitude of emission of a soft gluon with momentum $k^\mu = (\omega, \mathbf{k})$ off a hard parton i. The emission is described by the current

$$j^{\mu,b}(k) = \sum_{i=1}^{N_p} \frac{\omega\, p_i^\mu}{(kp_i)} T_i^b\,; \qquad \sum_{i=1}^{N_p} T_i^b \;=\; 0\,. \tag{2.1}$$

The latter equation represents conservation of the colour charge which guarantees current conservation, $k_\mu j^\mu = 0$. Squaring the current to compute the distribution we obtain

$$-j^2(k) \;=\; -2\sum_{i>j} T_i^b T_j^b \cdot w_{ij}(k)\,, \tag{2.2}$$

where w_{ij} is the "dipole antenna" distribution depending on parton angles,

$$w_{ij} = \frac{\omega^2\,(p_i p_j)}{(kp_i)(kp_j)} = \frac{\xi_{ij}}{\xi_i \xi_j}\,, \qquad \xi_{ij} = 1 - \cos\Theta_{\mathbf{p}_i,\mathbf{p}_j}\,, \qquad \xi_i = 1 - \cos\Theta_{\mathbf{p}_i,\mathbf{k}}\,. \tag{2.3}$$

2.1 Colour triviality of the $N_p = 2, 3$ cases and Sudakov form factors

For $N_p = 2, 3$ the products of colour generators that enter (2.2) are *proportional to the identity* in the colour space. Indeed, in the case of two quark jets in e^+e^- there is one dipole only and the corresponding colour factor is simply

$$-2T_1^b T_2^b = T_1^2 + T_2^2 = 2C_F\,.$$

For $N_p = 3$ (three jet $q\bar{q}g$ events) all colour factors can also be expressed as combinations of the Casimir operators T_i^2 since by virtue of the charge conservation in (2.1)

$$-2T_1T_2 \;=\; T_1^2 + T_2^2 - T_3^2 \qquad \text{(and cyclic)},$$

and one can write

$$-j^2(k) \;=\; T_1^2 \cdot W_{23}^{(1)}(k) + T_2^2 \cdot W_{13}^{(2)}(k) + T_3^2 \cdot W_{12}^{(3)}(k)\,, \tag{2.4}$$

where we have introduced the dipole combinations

$$W_{23}^{(1)} = w_{12} + w_{13} - w_{23}\,. \tag{2.5}$$

The essential property of the distribution (2.5) is that it is collinear singular *only* when $\mathbf{k} \parallel \mathbf{p}_1$. This singularity contributes proportional to the corresponding Casimir operator, in accord with general factorisation property. Integrating (2.5) over angles gives

$$\int \frac{d\Omega}{4\pi}\, W_{23}^{(1)} \;=\; \ln\frac{(p_1p_2)(p_1p_3)}{(p_2p_3)\, m^2} = \ln\frac{p_{t1,23}^2}{2\, m^2}\,, \tag{2.6}$$

with $p_{t1,23}$ transverse momentum of parton p_1 in the cms of the (p_2, p_3) pair, and m^2 the collinear cutoff. Exponentiating the soft gluon current squared (2.4) leads to the product of three Sudakov form factors. The collinear cutoff m disappears when the virtual and real contributions (to a collinear and infrared safe observable) are taken together, and gets replaced by the proper observable dependent scale $\mathcal{O}(Q_0)$, see examples in [8].

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For the system of $q(1)$, $\bar{q}(2)$ and a hard gluon $g(3)$ we have $T_1^2 = T_2^2 = C_F$, $T_3^2 = C_A$. Applying (2.6) to the full emission probability (2.4),

$$-\int \frac{d\Omega}{4\pi}\, j^2(k) = 2C_F \ln \frac{s_{q\bar{q}}}{2m^2} + N \ln \frac{p_{t3}^2}{2m^2}\,, \tag{2.7}$$

we see that the answer for the $q\bar{q}g$ system can be represented as the product of the two quark form factors at the scale $s_{q\bar{q}} = 2(p_1p_2)$ (cms energy of the $q\bar{q}$ pair) and the gluon form factor taken at the scale $p_{t3}^2 = 2(p_3p_1)(p_3p_2)/(p_1p_2)$ (invariant gluon transverse momentum) [9].

2.2 Large angle gluon radiation and the fifth form factor

Now we turn to processes involving four hard partons and consider $1+2 \to 3+4$ scattering characterised by kinematical variables

$$s = 2p_1p_2\,, \qquad -t = 2p_1p_3\,, \qquad -u = 2p_1p_4\,.$$

In what follows we will treat all three Mandelstam invariants as being of the same order of magnitude and discuss small angle scattering separately.

Soft gluon radiation off the four-parton ensemble is given by (2.1) for $N_p = 4$, with T_i^b quark, antiquark or gluon generator depending on the nature of participating parton i. We remark that within the convention (2.1) the generators are taken as if all partons were *incoming* (e.g., the actual colour charges of the outgoing partons 3 and 4 equal $-T_3^b$ and $-T_4^b$).

It is straightforward to verify that the sum of dipoles in (2.2) can be identically represented as

$$\begin{aligned} -\,j^2(k) = {} & T_1^2\, W_{34}^{(1)}(k) + T_2^2\, W_{34}^{(2)}(k) + T_3^2\, W_{12}^{(3)}(k) + T_4^2\, W_{12}^{(4)}(k) + \\ & + T_t^2 \cdot A_t(k) + T_u^2 \cdot A_u(k)\,. \end{aligned} \tag{2.8}$$

The first four terms form the product of the form factors attached to participating hard partons as before. Their angular integrals, see (2.6), are the same and give

$$\ln \frac{Q^2}{2m^2}\,, \qquad \text{with} \quad Q^2 = \frac{tu}{s} \;=\; s\,\sin^2\Theta_s\,, \tag{2.9}$$

which combination of invariants becomes the hard scale of the process common for all four Sudakov factors.

The last two terms in (2.8) give rise to the *fifth form factor* as discussed in the Introduction. The two operators T_t^2 and T_u^2 are the squared colour charges exchanged in the t and u channels of the scattering process,

$$T_t^2 = (T_3 + T_1)^2 = (T_2 + T_4)^2\,, \qquad T_u^2 = (T_4 + T_1)^2 = (T_2 + T_3)^2\,. \tag{2.10}$$

They do not commute and this is where the call for the colour diagonalisation programme comes from. The new angular dipole combinations that accompany these operators are given by

$$A_t = w_{12} + w_{34} - w_{13} - w_{24}\,, \quad A_u = w_{12} + w_{34} - w_{14} - w_{23}\,, \tag{2.11}$$

and, unlike the dipoles $W_{jk}^{(i)}$, are integrable in angles:

$$\int \frac{d\Omega}{4\pi} A_t(k) \;=\; 2\ln\frac{s}{-t}\,; \qquad \int \frac{d\Omega}{4\pi} A_u(k) \;=\; 2\ln\frac{s}{-u}\,. \tag{2.12}$$

Contrary to the first four DL contributions (that give rise to Sudakov form factors), the additional contribution originates from coherent gluon radiation at angles *larger than the cms scattering angle* Θ_s. Indeed, in the cms of colliding partons ($\xi_1+\xi_2=\xi_{12}=\xi_{34}=\xi_3+\xi_4=2$) we have

$$A_t = \frac{1}{\xi_1}\left[1-\frac{\xi_{13}-\xi_1}{\xi_3}\right] + \frac{1}{\xi_2}\left[1-\frac{\xi_{24}-\xi_2}{\xi_4}\right]. \tag{2.13}$$

Upon integration over the azimuth angle ϕ of the radiated gluon around the direction z of colliding parton momenta $\mathbf{p}_1=-\mathbf{p}_2$,

$$\int \frac{d\phi}{2\pi} A_t(k) = \frac{1}{\xi_1}\vartheta(\xi_1-\xi_{13}) + \frac{1}{\xi_2}\vartheta(\xi_2-\xi_{24})\,. \tag{2.14}$$

Thus, the gluon emission angle is limited *from below* by the t-channel scattering angle, $\xi_1 > \xi_{13} = \xi_{24}$. Analogously, in the T_u^2 term the lower limit is given by the u-channel scattering angle, $\xi_1 > \xi_{14}$. Polar angle integration then gives the logarithms of the ratio of Mandelstam invariants as stated in (2.12).

The rôle of coherent large angle gluon radiation driven by t-channel colour exchange was elucidated in [10] where comparison was made of the distributions of hadrons accompanying production of Higgs bosons via gluon–gluon and W^+W^- fusion.

2.3 Analysis of virtual corrections

Each parton channel has various colour channels:

$$a_1 + a_2 \;\to\; a_3 + a_4\,. \tag{2.15}$$

The hard scattering matrix element for a given channel is a function of four colour indices $\{a_i\}$ of participating partons: $a=1,\dots N$ for quark and $a=1,\dots N^2-1$ for gluon:

$$(M_0)_{a_2a_4}^{a_1a_3}\,; \quad \sigma_0 \;\propto\; \sum_{a_i} (M_0^\dagger)_{a_4a_2}^{a_3a_1}\,(M_0)_{a_2a_4}^{a_1a_3} \;=\; \mathrm{Tr}\left(M_0^\dagger\cdot M_0\right). \tag{2.16}$$

To address the problem of all-order analysis of soft radiation with single logarithmic accuracy, it suffices to study virtual corrections to hard patron matrix element M due to multiple soft gluons. Then, real production cross sections can be obtained simply by "cutting" the product $M^\dagger\cdot M$. Let k_V be the contribution to the observable V of the soft (real) gluon k. In the integration region $0<k_V\ll Q_0$ positive contribution to the cross section due to real production of gluons whose effect upon the observable is negligibly small cancels against negative virtual correction. After this standard real–virtual cancellation, the resulting distribution is determined by the virtual factor originating from the complementary integration region $Q_0<k_V<Q$.

The soft cross channel form factor $F_X(\tau_0, \{p_i\})$, the fifth form factor in (1.1), is expressed in terms of the matrix $M(\tau_0)$ that is obtained by considering virtual corrections to M_0 due to soft gluons with $k_t > Q_0$:

$$M_0 \implies M(\tau_0) = F_X(\tau_0) \cdot M_0\,, \qquad F_X(0) = 1\,. \tag{2.17}$$

For $Q_0 \sim Q$ we have $\tau_0 \ll 1$ and the matrix $M(\tau_0)$ reduces to M_0. The cross section then acquires the "soft factor" $\mathcal{S}_X(\tau_0)$ given in (1.4) and we may write

$$F_X^\dagger(\tau_0) \cdot F_X(\tau_0) \quad \Longrightarrow \quad \mathcal{S}_X(\tau_0)\,, \tag{2.18}$$

where F_X is a matrix in the colour space that will be dealt with in what follows.

The τ dependence of $M(\tau)$ can be extracted using the differential equation [3] that arises from the following iterative procedure. By virtue of soft gluon factorisation, the *softest* gluon k is emitted from the four external primary parton lines. Differentiating over its transverse momentum we obtain

$$\partial_\tau\, M(\tau) \;=\; G(\tau) \cdot M(\tau)\,, \qquad \tau = \int_{k_t}^{Q} \frac{dk_t'}{k_t'} \frac{\alpha_s(k_t')}{\pi}\,, \tag{2.19}$$

where $G(\tau)$ multiplies the matrix element $M(\tau)$ dressed by gluons that are *harder* than k_t. The soft anomalous dimension G is a colour matrix and is a function of s, t and u. It is not symmetric and is complex due to the s-channel gluon exchanges (Coulomb phases, [11], see also [3–6]).

Soft virtual corrections to the hard scattering matrix element can be split into two pieces: eikonal and Coulomb contributions.

Eikonal contribution. The first virtual contribution equals *minus one half* of the eikonal current squared (2.8) that we considered above in the context of *real* soft gluon radiation, and cancels it in the part of the phase space that is open for real gluons: $k_V \sim k_t \sim \omega < Q_0$. The colour trivial collinear logarithmic pieces in (2.8) are extracted and included into the exponents of four Sudakov form factors F_i at the hard scale Q given in (2.9). The remaining soft SL cross channel contributions, upon integration over gluon angles, give the soft anomalous dimension responsible for virtual suppression coming from the region $Q_0 < k_t < Q$ as stated in (1.2):

$$G^{\rm eik} = G_{\rm real} + G_{\rm virt}^{\rm eik} \;=\; -\left(T_t^2 \cdot \ln \frac{s}{-t} + T_u^2 \cdot \ln \frac{s}{-u} \right), \qquad Q_0 < k_t < Q\,. \tag{2.20}$$

Let us stress that this matrix is a constant in τ, the τ-dependence entering through the boundary in k_t.

Coulomb gluons. An additional virtual contribution arises when a soft virtual gluon connects two incoming or two outgoing partons. While in the eikonal contributions (both real and virtual) it was the gluon line that was put on-shell, the Coulomb correction is obtained by putting on-shell the two hard parton lines in the intermediate state. This contribution can be extracted by considering in (2.2) only the (p_1, p_2) and (p_3, p_4) interactions and replacing w_{12} and w_{34} by $i\pi$. The imaginary contributions present in the diagonal

(Sudakov) pieces in (2.8) give rise to Abelian Coulomb phase which fully cancels upon multiplication of the amplitude by the conjugate one. Imaginary part of the soft anomalous dimension reads

$$G_{\rm C} = i\pi\left(T_t^2 + T_u^2\right), \qquad 0 < k_t < Q\,; \tag{2.21}$$

it provides the amplitude with a non-Abelian Coulomb phase factor.

Combining real emission, virtual eikonal and Coulomb corrections, the final result for the soft anomalous dimension G becomes

$$G(\tau) = \Gamma\cdot\vartheta(\tau_0-\tau), \qquad \Gamma \equiv G^{\rm eik} + G_{\rm C} = -(T_t^2\,T + T_u^2\,U)\,; \tag{2.22}$$

$$T = \ln\frac{s}{-t} - i\pi\,, \quad U = \ln\frac{s}{-u} - i\pi\,, \tag{2.23}$$

for $Q_0 < k_t < Q$, and

$$G(\tau) = \Gamma_C\cdot\vartheta(\tau-\tau_0), \qquad \Gamma_C \equiv G_{\rm C} = i\pi\left(T_t^2 + T_u^2\right), \tag{2.24}$$

for $0 < k_t < Q_0$.

Exponentiation. The evolution equation (2.19) has to be integrated over τ from 0 ($k_t = Q$) up to $\Lambda\to\infty$ ($k_t = 0$). The formal solution is given by a τ-ordered exponent

$$M(\tau_0) \;=\; P_\tau \exp\left\{\int_0^\Lambda d\tau\, G(\tau)\right\}\cdot M_0\,. \tag{2.25}$$

Since $G(\tau)$ assumes (different) *constant* values for $0 < \tau < \tau_0$ and $\tau_0 < \tau < \infty$, using (2.22)–(2.24) we obtain

$$M(\tau_0) \;=\; e^{(\Lambda-\tau_0)\Gamma_C}\, e^{\tau_0\,\Gamma}\cdot M_0\,. \tag{2.26}$$

Cancellation of divergent Coulomb phase. For $\Lambda\to\infty$ ($k_t\to 0$) the first factor in (2.26) diverges. However, since exchanging a gluon between two incoming (or outgoing) partons obviously does not affect their overall colour state, the sum of the two matrices $T_t^2 + T_u^2$ is necessarily *diagonal* in s-channel colour,

$$T_t^2 + T_u^2 \;=\; -T_s^2 + \sum_{i=1}^{4} T_i^2\,, \qquad T_s = T_1 + T_2 = -(T_3+T_4)\,. \tag{2.27}$$

Therefore the Coulomb matrix Γ_C in (2.24) is anti-Hermitian. Thus the first factor in (2.26) becomes a unitary matrix and cancels upon multiplication with the conjugate amplitude in (1.4). Therefore, in the calculation of physical distributions one can effectively neglect the divergent Coulomb phase factor in (2.26) and use

$$M(\tau_0) \;=\; e^{\tau_0\,\Gamma}\cdot M(0)\,, \qquad \Gamma = -\left(T_t^2\cdot T + T_u^2\cdot U\right). \tag{2.28}$$

Let us stress that cancellation of the non-Abelian Coulomb phase is only partial. An imaginary Coulomb contribution coming from the virtual gluon momentum region $Q_0 < k_t < Q$ is still present and enters Γ through complex logarithms T and U defined in (2.23).

2.4 Colour structure

To evaluate the matrices T_t^2 and T_u^2 in (2.28) we turn to the analysis of the colour structure of the process in various channels.

The operator T_t^2 is a number in a given colour state of the t-channel parton pair, (p_1, p_3) and/or (p_2, p_4). Therefore, in the t-channel projector basis it is the diagonal matrix of Casimirs. Similarly, T_u^2 is diagonal in the u-channel projector basis (p_1, p_4) or (p_2, p_3).

For calculation of the s-channel observables it is natural, however, to describe colour states from the s-channel point of view. We will use the s-channel basis of projectors $\mathcal{P}_\alpha$ onto irreducible $SU(N)$ representations that are present in the colour space of two incoming partons (p_1, p_2) or, equivalently, outgoing (p_3, p_4). The completeness relation reads

$$\left(\mathbb{1}\right)^{a_1 a_3}_{a_2 a_4} = \delta^{a_1, a_3} \delta_{a_2, a_4}\,, \qquad \mathbb{1} = \sum_{\alpha=1}^{\mathcal{N}} \mathcal{P}_\alpha\,, \qquad \mathrm{Tr}(\mathcal{P}_\alpha) = K_\alpha\,. \tag{2.29}$$

Here K_α is the dimension of the representation α, and $\mathcal{N}$ the number of irreducible representations involved. The sum over all K_α equals the total number of colours states of the system of two incoming (or outgoing) partons. The matrix element can be expressed as a sum over colour projectors

$$M_0 = \sum_{\alpha=1}^{\mathcal{N}} m_{0\alpha}\, \mathcal{P}_\alpha\,, \qquad M(\tau) = \sum_{\alpha=1}^{\mathcal{N}} m_\alpha(\tau)\, \mathcal{P}_\alpha\,. \tag{2.30}$$

In order to compute $T_{t,u}^2$ we need to know the transition matrices connecting s- and t-/u-channels projectors. Introducing

$$\mathcal{P}^{(t)} = K_{ts} \cdot \mathcal{P}\,, \quad \mathcal{P} = K_{st} \cdot \mathcal{P}^{(t)}\,; \qquad K_{st} = (K_{ts})^{-1}\,; \tag{2.31a}$$

$$\mathcal{P}^{(u)} = K_{us} \cdot \mathcal{P}, \quad \mathcal{P} = K_{su} \cdot \mathcal{P}^{(u)}; \qquad K_{su} = (K_{us})^{-1}\,, \tag{2.31b}$$

we have

$$\left(T_t^2\right)_{\alpha\beta} = \sum_\rho (K_{st})_{\alpha\rho}\, c_\rho^{(t)}\, (K_{ts})_{\rho\beta}\,, \tag{2.32}$$

where the indices α and β mark irreducible representations of the colour group of the pair of incoming partons (s-channel) and ρ — representations of the t-channel pair. In particular, $c_\rho^{(t)}$ stand for the Casimir operator of the t-channel representation ρ.

In general, K are not necessarily square matrices. For example, for the process $q\bar{q} \to gg$ we have two colour states in the s-channel, **1** and **8**, while there are three in the t-channel: **3**, $\bar{\mathbf{6}}$ and **15**.

We conclude this section devoted to general discussion of the physics and technical ingredients of the analysis of the fifth form factor by presenting the final general expression which holds for scattering of arbitrary colour objects:

$$F_X(\tau_0) \;=\; \exp\left\{-\tau_0 \left(T_t^2 \cdot T + T_u^2 \cdot U\right)\right\}, \tag{2.33}$$

where the cross-channel squared colour charge matrices T_t^2 and T_u^2 have to be computed using (2.32). We will demonstrate this computation on the concrete case of gluon–gluon scattering.

3. Gluon-gluon scattering

The $gg \to gg$ amplitude has $(N^2-1)^4$ colour indices. We start by introducing the s-channel projector basis for this process.

3.1 Irreducible representations and projectors

The colour state of two gluons can be characterised in terms of irreducible representations. In $SU(3)$ we have

$$\mathbf{glue} \otimes \mathbf{glue} = \mathbf{8_a} + \mathbf{10} + \mathbf{1} + \mathbf{8_s} + \mathbf{27}\,, \tag{3.1}$$

where $\mathbf{8_a}$ and $\mathbf{10}$ mark *antisymmetric* representations (octet and the sum of the decuplet and anti-decuplet) and three *symmetric* ones are the singlet ($\mathbf{1}$), octet ($\mathbf{8_s}$) and the high symmetric tensor representation ($\mathbf{27}$) with corresponding dimension. In the general case of $SU(N)$ (with $N > 3$) we have an additional *symmetric* representation (which we mark $\mathbf{0}$):

$$\mathbf{glue} \otimes \mathbf{glue} = \mathbf{8_a} + \mathbf{10} + \mathbf{1} + \mathbf{8_s} + \mathbf{27} + \mathbf{0}\,. \tag{3.2}$$

Thus we will keep using the $SU(3)$ motivated names in spite of the fact that the dimensions of corresponding representations are actually different from 8, 2×10, etc.:

$$\begin{aligned} K_{\mathbf{1}} &= 1\,, \qquad K_{\mathbf{a}} = K_{\mathbf{s}} = N^2-1\,, \qquad K_{\mathbf{10}} = 2\times\frac{(N^2-1)(N^2-4)}{4}\,,\\ K_{\mathbf{27}} &= \frac{N^2(N-1)(N+3)}{4}\,, \qquad K_{\mathbf{0}} = \frac{N^2(N+1)(N-3)}{4}\,. \end{aligned} \tag{3.3}$$

(Mark that for $N = 3$ we have indeed $K_{\mathbf{0}} = 0$.) It is straightforward to check that the sum of the dimensions (3.3) gives $(N^2-1)^2$ as expected.

As was explained above, it is convenient to work in the colour space of hard gluon scattering in the s-channel. We construct the basis of s-channel projectors $\mathcal{P}_\alpha$ ordered as follows:

$$\alpha = \{\mathbf{8_a}, \mathbf{10}, \mathbf{1}, \mathbf{8_s}, \mathbf{27}, \mathbf{0}\}\,. \tag{3.4}$$

The projectors are explicitly constructed in appendix A. They satisfy the completeness relation

$$\sum_{\alpha=1}^{6} \mathcal{P}_\alpha = \mathcal{P}_{\mathbf{a}} + \mathcal{P}_{\mathbf{10}} + \mathcal{P}_{\mathbf{1}} + \mathcal{P}_{\mathbf{s}} + \mathcal{P}_{\mathbf{27}} + \mathcal{P}_{\mathbf{0}} = \mathbb{1}\,; \qquad (\mathbb{1})^{a_1a_3}_{a_2a_4} = \delta^{a_1,a_3}\delta_{a_2,a_4}\,. \tag{3.5}$$

Dimension of a given representation (3.3) can be calculated by taking trace of the corresponding projector:

$$\mathrm{Tr}\,(\mathcal{P}_\alpha) \equiv \sum_{a_1,a_2} (\mathcal{P}_\alpha)^{a_1\,a_1}_{a_2\,a_2} \;=\; K_\alpha\,. \tag{3.6}$$

Casimir operators of all six representations are calculated in appendix A.2. In our basis (3.4) the diagonal matrix of Casimir operators reads

$$(T^a)^2_{\alpha\beta} \;=\; (C_2)_{\alpha\beta} = \delta_{\alpha\beta}\cdot c_\alpha\,, \qquad c_\alpha = \{N,\, 2N,\, 0,\, N,\, 2(N+1),\, 2(N-1)\}. \tag{3.7}$$

This matrix enters the expression (2.32) for the anomalous dimension Γ. Matrices K_{ts} and K_{us} that rotate the s-channel projector basis into t- and u-channels are calculated in appendix A.3.

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3.2 Diagonalisation of the anomalous dimension matrix

We shall represent the anomalous dimension matrix as

$$\Gamma \;=\; -N(T+U)\cdot\mathcal{Q}\,, \tag{3.8}$$

where the matrix $\mathcal{Q}$ depends on the ratio of the logarithmic variables

$$b \;\equiv\; \frac{T-U}{T+U}\,, \tag{3.9}$$

$$\mathcal{Q} \;=\; \begin{pmatrix} \frac{3}{2} & 0 & -2b & -\frac{1}{2}b & -\frac{2}{N^2}b & -\frac{2}{N^2}b \\ 0 & 1 & 0 & -b & -\frac{(N+1)(N-2)}{N^2}b & -\frac{(N-1)(N+2)}{N^2}b \\ -\frac{2}{N^2-1}b & 0 & 2 & 0 & 0 & 0 \\ -\frac{1}{2}b & -\frac{2}{N^2-4}b & 0 & \frac{3}{2} & 0 & 0 \\ -\frac{N+3}{2(N+1)}b & -\frac{N+3}{2(N+2)}b & 0 & 0 & \frac{N-1}{N} & 0 \\ -\frac{N-3}{2(N-1)}b & -\frac{N-3}{2(N-2)}b & 0 & 0 & 0 & \frac{N+1}{N} \end{pmatrix} \tag{3.10}$$

It is worth observing that the matrix elements of the states **27** and **0** (two last rows and columns) are formally related by the operation $N \to -N$.

3.2.1 Eigenvalues and eigenvectors

Six eigenstates of $\mathcal{Q}$ naturally split into two groups of three.

The first three. The first three eigenvalues are N-independent:

$$E_1 = 1\,, \qquad E_2 = \frac{3-b}{2}\,, \qquad E_3 = \frac{3+b}{2}\,. \tag{3.11}$$

The eigenvectors $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ corresponding to these eigenvalues are

$$\mathcal{V}_{1,2,3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{4b}{N^2-4} \\ -\frac{b\,N(N+3)}{2(N+2)} \\ \frac{b\,N(N-3)}{2(N-2)} \end{bmatrix} \begin{bmatrix} 1+b \\ -2b \\ \frac{4b}{N^2-1} \\ 1+\frac{b\,(N^2-12)}{N^2-4} \\ -\frac{b\,N(N+3)}{(N+1)(N+2)} \\ -\frac{b\,N(N-3)}{(N-1)(N-2)} \end{bmatrix} \begin{bmatrix} -1+b \\ -2b \\ -\frac{4b}{N^2-1} \\ 1-\frac{b\,(N^2-12)}{N^2-4} \\ \frac{b\,N(N+3)}{(N+1)(N+2)} \\ \frac{b\,N(N-3)}{(N-1)(N-2)} \end{bmatrix}. \tag{3.12}$$

The states 2 and 3 are related by the crossing transformation $t \leftrightarrow u$. In particular, the eigenvector $\mathcal{V}_3$ is obtained from $\mathcal{V}_2$ by $b \to -b$ and changing the sign of the antisymmetric projector components (first two rows).

We also remark that N enters the $\mathcal{P}_{\mathbf{27}}$ and $\mathcal{P}_{\mathbf{0}}$ components with the opposite sign thus reflecting the symmetry of the matrix elements of $\mathcal{Q}$.

The last three. The last three eigenvalues solve the cubic equation

$$\left[E_i - \frac{4}{3}\right]^3 - \frac{(1+3b^2)(1+3x^2)}{3}\left[E_i - \frac{4}{3}\right] - \frac{2(1-9b^2)(1-9x^2)}{27} \;=\; 0\,, \tag{3.13}$$

where we have introduced the notation

$$x \;=\; \frac{1}{N}\,. \tag{3.14}$$

Solutions can be parametrised as follows:

$$E_{4,5,6} \;=\; \frac{4}{3}\left(1 + \frac{\sqrt{(1+3b^2)(1+3x^2)}}{2}\cos\left[\frac{\phi+2k\pi}{3}\right]\right); \quad k=0,\,1,\,2\,, \tag{3.15a}$$

where ϕ is given by

$$\cos\phi = R\,, \qquad R = \frac{(1-9b^2)(1-9x^2)}{[(1+3b^2)(1+3x^2)]^{\frac{3}{2}}}\,. \tag{3.15b}$$

The last three eigenvectors are

$$\mathcal{V}_{4,5,6} = \begin{bmatrix} -\frac{4}{N^2}\,(E_i-1)\,b \\ -\frac{N^2-4}{N^2}\,(E_i-2)\,b \\ \frac{1}{N^2-1}\left[\left(E_i - \frac{N-1}{N}\right)\left(E_i - \frac{N+1}{N}\right) - \frac{N^2-5}{N^2}\,b^2\right] \\ \frac{4}{N^2}\,b^2 \\ \frac{N}{N+1}\left[\frac{N+2}{2\,N}(E_i-2)\left(E_i - \frac{N+1}{N}\right) - 2\,b^2\right] \\ \frac{N}{N-1}\left[\frac{N-2}{2\,N}(E_i-2)\left(E_i - \frac{N-1}{N}\right) - 2\,b^2\right] \end{bmatrix}, \tag{3.16}$$

with E_i the corresponding energy eigenvalue, $i = 4,5,6$.

Vectors $\mathcal{V}_i$ are orthogonal with respect to the scalar product defined by the metric tensor $W_{\alpha\beta} = K_\alpha \delta_{\alpha\beta}$,

$$\langle \mathcal{V}_i |\, W^{-1}\, | \mathcal{V}_k \rangle \;=\; 0\,, \quad i \neq k\,.$$

3.2.2 Strange symmetry

We note an unexpected mysterious property of the equation (3.13) for the eigenvalues of the soft anomalous dimension matrix which is symmetric with respect to

$$b = \frac{T-U}{T+U} \quad \Longleftrightarrow \quad x = \frac{1}{N}\,, \tag{3.17}$$

the transformation that interchanges parameters characterising external (scattering angle) and internal (colour group) degrees of freedom.

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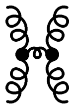
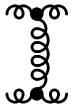
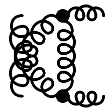
3.3 Hard matrix element

The colour structure of the hard gluon scattering matrix element can be represented in terms of the s-, t- and u-channel projectors (see appendix A)

$$\mathcal{P}_{\mathbf{a}} = \frac{1}{N}$$  , $$\mathcal{P}_{\mathbf{a}}^{(t)} = \frac{1}{N}$$  , $$\mathcal{P}_{\mathbf{a}}^{(u)} = \frac{1}{N}$$  . (3.18)

We have

$$M_0 = N\left(m_s\,\mathcal{P}_{\mathbf{a}} + m_t\,\mathcal{P}_{\mathbf{a}}^{(t)} + m_u\,\mathcal{P}_{\mathbf{a}}^{(u)} \right), \tag{3.19}$$

where we have included into m_λ ($\lambda = s, t, u$) the one-gluon exchange diagram in the λ-channel together with the piece of the four-gluon vertex contribution that has the same colour structure. The t- and u-channel projectors $P_{\mathbf{a}}^{(t)}$ and $P_{\mathbf{a}}^{(u)}$ can be expressed in terms of the s-channel ones introduced in (3.5) (see appendix A.3) as follows:

$$\begin{aligned}
\mathcal{P}_{\mathbf{a}}^{(t)} &= \frac{1}{2}\mathcal{P}_{\mathbf{a}} + \mathcal{P}_{\mathbf{1}} + \frac{1}{2}\mathcal{P}_{\mathbf{s}} - \frac{1}{N}\mathcal{P}_{\mathbf{27}} + \frac{1}{N}\mathcal{P}_{\mathbf{0}}\,, \\
\mathcal{P}_{\mathbf{a}}^{(u)} &= -\frac{1}{2}\mathcal{P}_{\mathbf{a}} + \mathcal{P}_{\mathbf{1}} + \frac{1}{2}\mathcal{P}_{\mathbf{s}} - \frac{1}{N}\mathcal{P}_{\mathbf{27}} + \frac{1}{N}\mathcal{P}_{\mathbf{0}}\,.
\end{aligned} \tag{3.20}$$

We obtain

$$M_0 = N\left[\frac{M_a}{2}\mathcal{P}_{\mathbf{a}} + M_s\left(\mathcal{P}_{\mathbf{1}} + \frac{1}{2}\mathcal{P}_{\mathbf{s}} - \frac{1}{N}\mathcal{P}_{\mathbf{27}} + \frac{1}{N}\mathcal{P}_{\mathbf{0}} \right) \right], \tag{3.21}$$

where M_a and M_s are, respectively, the parts of the matrix element antisymmetric and symmetric with respect to exchange of gluons in the s-channel:

$$M_a = 2m_s + m_t - m_u\,, \qquad M_s = m_t + m_u\,. \tag{3.22}$$

It is worthwhile to notice that these amplitudes are separately gauge invariant. Squaring the matrix element (3.21) gives

$$|M_0|^2 = N^2\left[\frac{M_a^2}{4}\cdot\mathcal{P}_{\mathbf{a}} + M_s^2\cdot\left\{ \mathcal{P}_{\mathbf{1}} + \frac{1}{4}\mathcal{P}_{\mathbf{s}} + \frac{1}{N^2}\mathcal{P}_{\mathbf{27}} + \frac{1}{N^2}\mathcal{P}_{\mathbf{0}} \right\} \right]. \tag{3.23}$$

Here

$$M_a^2 \equiv (2m_s + m_t - m_u)^2 = 9 - \frac{st}{u^2} - \frac{us}{t^2} - \frac{4tu}{s^2} - \frac{3s^2}{tu}, \tag{3.24}$$

$$M_s^2 \equiv (m_t + m_u)^2 = 1 - \frac{st}{u^2} - \frac{us}{t^2} + \frac{s^2}{tu}, \tag{3.25}$$

where we have used the known Lorentz matrix elements.

The total scattering cross section is proportional to the colour trace of the squared matrix element (3.23):

$$\sigma_0 \equiv \mathrm{Tr}\big(M_0^2\big) = N^2(N^2-1)\frac{M_a^2 + 3M_s^2}{4} = N^2(N^2-1)\left\{ 3 - \frac{tu}{s^2} - \frac{us}{t^2} - \frac{st}{u^2} \right\}, \tag{3.26}$$

where we have used the kinematical relation

$$\frac{s^2}{tu} + \frac{t^2}{su} + \frac{u^2}{st} = 3\,.$$

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Let us mention another elegant representation for the colour summed cross section,

$$\sigma_0 = \frac{N^2}{2}(N^2-1)\left[(m_t+m_u)^2 + (m_u-m_s)^2 + (m_s+m_t)^2\right]. \tag{3.27}$$

Here the first term in square brackets is given in (3.24) and the other two can be obtained from it by simple crossing, that is by replacing $s \leftrightarrow t$ and $s \leftrightarrow u$, respectively.

3.4 Squaring dressed matrix element

Now we are in a position to construct the dressed matrix element according to (2.28) and evaluate the cross section. Expressing the eigenvectors (3.12), (3.16) as

$$\mathcal{V}_\kappa = (Z\cdot\mathcal{P})_\kappa = \sum_\alpha Z_\kappa^\alpha\,\mathcal{P}_\alpha\,, \qquad \mathcal{P}_\alpha = (Z^{-1}\cdot\mathcal{V})_\alpha = \sum_\kappa \left(Z^{-1}\right)_\alpha^\kappa \mathcal{V}_\kappa\,, \tag{3.28}$$

for the evolution exponent we have

$$e^{\Gamma\tau}M_0 \;=\; \sum_\beta m^\beta(\tau)\mathcal{P}_\beta\,, \tag{3.29}$$

where we have introduced

$$m^\beta(\tau) \;=\; \sum_\alpha m^{(0)\alpha}\,\sum_\kappa (Z^{-1})_\alpha^\kappa\cdot e^{-N(T+U)\tau\,E_\kappa}\cdot Z_\kappa^\beta\,. \tag{3.30}$$

The soft factor $\mathcal{S}_X$ becomes

$$\mathcal{S}_X(\tau) \;=\; \sigma_0^{-1}\,\mathrm{Tr}\left(M^\dagger(\tau)\cdot M(\tau)\right) = \sigma_0^{-1}\sum_\beta \left|m^\beta(\tau)\right|^2\cdot K_\beta\,. \tag{3.31}$$

3.5 Special cases

Now we turn to the discussion of special cases in which the answer is relatively simple and can be given explicitly.

3.5.1 Scattering at $90^{\rm o}$

Consider first the simple case of $b=0$ ($t=u$) which corresponds to 90 degree scattering. Here $\mathcal{Q}$ is diagonal so that the s-channel projectors $\mathcal{P}_\alpha$ become eigenvectors whose eigenvalues are just the corresponding diagonal elements of $\mathcal{Q}$:

$$E_\kappa = \left\{1,\,\frac{3}{2},\,\frac{3}{2},\,2,\,\frac{N-1}{N},\,\frac{N+1}{N}\right\}, \tag{3.32}$$

$$\mathcal{V}_\kappa \propto \{\mathcal{P}_{\mathbf{10}},\,\mathcal{P}_{\mathbf{s}}+\mathcal{P}_a,\,\mathcal{P}_{\mathbf{s}}-\mathcal{P}_a,\,\mathcal{P}_{\mathbf{1}},\,\mathcal{P}_{\mathbf{27}},\,\mathcal{P}_{\mathbf{0}}\}\,. \tag{3.33}$$

To present the answer for the soft factor $\mathcal{S}_X$ it is convenient to define the suppression factors

$$\chi_t(\tau) \;=\; \exp\left\{-2N\tau\cdot\ln\frac{s}{-t}\right\}, \quad \chi_u(\tau) \;=\; \exp\left\{-2N\tau\cdot\ln\frac{s}{-u}\right\}. \tag{3.34}$$

In 90^o scattering kinematics we have $\mathrm{Re}\,T = \mathrm{Re}\,U = \ln 2$ and

$$\chi_t = \chi_u \;=\; \chi(\tau) \;=\; \exp\left\{-2N\tau\ln 2\right\}, \tag{3.35}$$

and we get

$$\mathcal{S}_X(\tau) = \frac{\chi^2}{3}\left[\frac{4}{N^2-1}\,\chi^2 + \chi + \frac{N-3}{N-1}\,\chi^{\frac{2}{N}} + \frac{N+3}{N+1}\,\chi^{-\frac{2}{N}}\right]. \tag{3.36}$$

We check that for $\tau = 0$, $\chi = 1$, (3.36) gives indeed $\mathcal{S}_X(0) = 1$ as it should.

The cubic equation (3.15b) trivialises and can be solved explicitly for $b=0$:

$$\cos\phi = \frac{1-9x^2}{(1+3x^2)^{3/2}}, \quad \Longrightarrow \quad \cos\frac{\phi}{3} = \frac{1}{\sqrt{1+3x^2}},$$
$$\cos\left(\frac{\phi}{3} \pm \frac{2\pi}{3}\right) = -\frac{1\,\pm 3x}{2\sqrt{1+3x^2}}. \tag{3.37}$$

Substituting (3.37) into (3.15) gives the last three energy levels in (3.32).

3.5.2 $N \to \infty$ limit

By virtue of the weird symmetry (3.17), the large-N limit ($x \to 0$) is related with the 90^o scattering case considered above: the N-dependent energy levels 4, 5, 6 can be obtained from (3.32) simply by replacing $x \to b$,

$$\cos\frac{\phi}{3} = \frac{1}{\sqrt{1+3b^2}}, \qquad \cos\left(\frac{\phi}{3} \pm \frac{2\pi}{3}\right) = -\frac{1\,\pm 3b}{2\sqrt{1+3b^2}}. \tag{3.38}$$

The energy levels are as follows:

$$E_1 = 1, \qquad E_2 = \frac{3-b}{2}, \qquad E_3 = \frac{3+b}{2}, \qquad E_4 = 2, \qquad E_5 = 1-b, \qquad E_6 = 1+b.$$

The weird symmetry does not extend upon the eigenvectors so that they have to be derived anew:

$$\mathcal{V}_{1,\ldots 6} = \begin{bmatrix} 0\\0\\0\\0\\1\\-1 \end{bmatrix} \begin{bmatrix} 1+b\\-2b\\0\\1+b\\-b\\-b \end{bmatrix} \begin{bmatrix} 1-b\\2b\\0\\-1+b\\-b\\-b \end{bmatrix} \begin{bmatrix} -4(1-b^2)\\-8b^3\\(1-b^2)^2\\4(1-b^2)\\2b^2(1+b^2)\\2b^2(1+b^2) \end{bmatrix} \begin{bmatrix} 0\\2\\0\\0\\1\\1 \end{bmatrix} \begin{bmatrix} 0\\-2\\0\\0\\1\\1 \end{bmatrix}. \tag{3.39}$$

The soft factor becomes

$$\mathcal{S}_X \simeq \frac{\chi_t\,\chi_u}{2\,(M_a^2+3M_s^2)}\left[4\,M_s^2 + (M_a-M_s)^2\,\chi_t + (M_a+M_s)^2\,\chi_u\right]. \tag{3.40}$$

3.5.3 Regge limit

In the case $b \to \pm 1$ (small angle scattering) (3.15b) can be solved explicitly too, for arbitrary N:

$$\cos\phi = -\frac{1-9x^2}{(1+3x^2)^{3/2}}, \quad \Longrightarrow \quad \cos\frac{\phi}{3} = \frac{-1}{\sqrt{1+3x^2}}, \tag{3.41}$$

$$\cos\left(\frac{\phi}{3} \pm \frac{2\pi}{3}\right) = \frac{1\,\mp 3x}{2\sqrt{1+3x^2}}. \tag{3.42}$$

Substituting (3.41) into (3.15) and invoking (3.11) produces the set of eigenvalues

$$\{E_\kappa\} \;=\; \{1,\, 1,\, 2;\, 0,\, 2(1-x), 2(1+x)\}\,. \tag{3.43}$$

In what follows we consider the forward scattering case, $b \to +1$.

For $|t| \ll s \simeq |u|$ we have $T \simeq \ln \frac{s}{-t} \gg |U| \simeq \pi$. Neglecting the finite phase, in the logarithmic approximation in T the soft matrix (2.33) becomes diagonal in the t-channel basis

$$F_X(\tau) = e^{\tau\Gamma} \;\simeq\; K_{st}\, e^{-C_2\cdot\tau\,\ln(s/t)}\, K_{ts}, \tag{3.44}$$

where C_2 is the diagonal matrix of the Casimirs (3.7) and the matrices K are given in appendix A.3. These exponents describe *reggeization* of six possible t-channel colour states. Indeed, the energy levels $N\!\cdot\! E_\kappa$ in (3.43) equal the Casimir operators. Cast in the canonical t-channel order (3.4), the eigenvalues read

$$c_\alpha \;=\; \{N,\, 2N,\, 0,\, N, 2(N+1),\, 2(N-1)\} \;=\; N\cdot\{E_1,\, E_3,\, E_4,\, E_2,\, E_6,\, E_5\}\,.$$

The eigenvectors corresponding to the energies (3.43) become, accordingly, pure t-channel projector states,

$$\begin{aligned} &\mathcal{V}_1 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{a}}\,, &&\mathcal{V}_2 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{s}}\,, &&\mathcal{V}_3 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{10}}\,,\\ &\mathcal{V}_4 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{1}}\,, &&\mathcal{V}_5 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{0}}, &&\mathcal{V}_6 = K_{st}\cdot\mathcal{P}^{(t)}_{\mathbf{27}}, \end{aligned} \tag{3.45}$$

and are given, correspondingly, by the *columns* # 1, 4, 2, 3, 6 and 5 of the re-projection matrix K_{ts} (A.28). In our case of the order α_{s} matrix element, (3.21) in the $t \to 0$ limit reduces to *only one state* namely, that of the asymmetric t-channel octet. Indeed, for $M_s = M_a \simeq m_t$ we have

$$M_0 \;\simeq\; N\cdot\mathcal{P}^{(t)}_{\mathbf{a}}\,, \qquad F_X\cdot M_0 = \left(\frac{s}{t}\right)^{-N\tau}\cdot M_0\,,$$

giving

$$S(\tau) \;=\; \chi_t(\tau) \;=\; \left(\frac{s}{t}\right)^{-2N\tau}\,, \tag{3.46}$$

which exponent coincides with the (twice) Regge trajectory of the gluon exchanged in the t-channel.

3.5.4 $N = 3$

In $SU(3)$ the representation $\mathbf{0}$ has zero weight, $K_{\mathbf{0}} = 0$, and the projector $\mathcal{P}_{\mathbf{0}}$ does not contribute. The projector basis reduces to five states: $\alpha = \{\mathbf{a}, \mathbf{10}, \mathbf{1}, \mathbf{s}, \mathbf{27}\}$. The reduced anomalous dimension matrix $\mathcal{Q}$ is obtained from (3.10) by setting $N\!=\!3$ and removing the last row and column. The first three eigenvalues $E_{1,2,3}$ are N-independent and given by (3.11). The other three $E_{4,5,6}$ are easy to obtain from (3.15) where $\phi = \pi/2$. Dropping the eigenvalue $E_6 = 4/3$ attached to the fake $\mathbf{0}$ state, we have five energy levels

$$E_\kappa = \left\{1, \frac{3-b}{2}, \frac{3+b}{2}, \frac{4+2\sqrt{1+3b^2}}{3}, \frac{4-2\sqrt{1+3b^2}}{3}\right\}. \tag{3.47}$$

The corresponding eigenvectors $\mathcal{V}_{1,\dots 5}$ read

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{4b}{5} \\ -\frac{9b}{5} \end{bmatrix} \begin{bmatrix} b{+}1 \\ -2b \\ \frac{b}{2} \\ 1{-}\frac{3b}{5} \\ -\frac{9b}{10} \end{bmatrix} \begin{bmatrix} b{-}1 \\ -2b \\ -\frac{b}{2} \\ 1{+}\frac{3b}{5} \\ \frac{9b}{10} \end{bmatrix} \begin{bmatrix} -\frac{4b\left(1{+}2\sqrt{1{+}3b^2}\right)}{27} \\ \frac{10b\left(1{-}\sqrt{1{+}3b^2}\right)}{27} \\ \frac{1{+}\sqrt{1{+}3b^2}}{18}+\frac{b^2}{9} \\ \frac{4\,b^2}{9} \\ \frac{5\left(1{-}\sqrt{1{+}3b^2}\right)}{18}+\frac{2b^2}{3} \end{bmatrix} \begin{bmatrix} -\frac{4b\left(1{-}2\sqrt{1{+}3b^2}\right)}{27} \\ \frac{10b\left(1{+}\sqrt{1{+}3b^2}\right)}{27} \\ \frac{1{-}\sqrt{1{+}3b^2}}{18}+\frac{b^2}{9} \\ \frac{4\,b^2}{9} \\ \frac{5\left(1{+}\sqrt{1{+}3b^2}\right)}{18}+\frac{2b^2}{3} \end{bmatrix}$$

4. Conclusions

In this paper we considered the soft SL factor $\mathcal{S}_X$ defined in (1.4) that enters the general representation (1.3) for two scale QCD observables in hadron–hadron collisions. Being collinear safe, this factor is driven by emission of soft gluons at large angles, see (2.14). For *global* observables the problem reduces to the analysis of soft radiation off the primary hard partons p_i $(i = 1, \dots 4)$ only, and essentially trivialises in spite of remaining non-Abelian. Indeed, such accompanying radiation has classical nature described by eikonal currents and Coulomb phase effects. Virtual and real *eikonal* contributions due to such gluons fully cancel in the phase space region $k_t < Q_0$, while the Coulomb contributions reduce to an (infinite) colour matrix phase that cancels in the distributions. The net result is the virtual eikonal suppression, accompanied by finite non-Abelian Coulomb phase, due to the complementary momentum region $Q_0 < k_t < Q$.

Our first result is the general simple expression for virtual dressing of the scattering matrix element, in terms of colour charges (Casimir operators) of the cross-channel colour exchanges, T_t^2 and T_u^2:

$$\frac{1}{2}\int (dk)\, j^2(k) \;=\; \frac{1}{2}\sum_{i=1}^{4} T_i^2\cdot \mathcal{R}_i \;+\; T_t^2\cdot T + T_u^2\cdot U \qquad \left(T = \ln\frac{s}{t}\,,\quad U = \ln\frac{s}{u}\right). \tag{4.1}$$

Here $\mathcal{R}_i$ are the (colour-trivial) "radiators" that accommodate collinear singularities and participate in forming the collinear factor in (1.3). The last two terms form the soft anomalous dimension matrix Γ that determines $\mathcal{S}_X$ as a function of the SL variable τ (1.2). Our anomalous dimension differs from the one introduced in [4, 5] by a piece proportional to the unit matrix. In our approach, this piece is absorbed into collinear parton radiators. It participates in determining the precise scales of the DL form factors in (1.3) in terms of a combination of angular integrated soft dipoles $W_{ij}^{(\ell)}$ each of which produces the same scale $Q^2 = tu/s$, see (2.9).

The matrices T_t^2 and T_u^2 do not commute. We found it convenient to work in the colour basis of s-channel projectors where each of them can be easily found with use of the re-projection matrices (2.31),

$$T_t^2 = K_{st}\, C_2^{(t)}\, K_{ts}\,, \qquad T_u^2 = K_{su}\, C_2^{(u)}\, K_{us}\,, \tag{4.2}$$

with C_2 the diagonal matrix of Casimir operators of all irreducible representations present in the t (u) channel.

Using the s-channel language makes the treatment and understanding of the results more transparent. The graphical colour projection technique presented in the appendix allowed us to avoid using the over-complete Chan-Paton basis and largely simplified the analysis. In particular, the calculation of the key ingredients (4.2) of the anomalous dimension becomes very simple since knowing the transformation matrices K the problem reduces to the Casimirs (for gluon-gluon scattering the Casimirs are given in (3.7)).

Another advantage of the representation for Γ in (2.28) in terms of cross-channel charges (4.2) is trivialisation of the analysis of the Regge behaviour. In the case of small angle scattering one term dominates, $T \gg U$ (forward scattering) or $U \gg T$ (backward), and the anomalous dimension Γ becomes diagonal in the corresponding channel so that the problem becomes essentially Abelian. Resulting exponents are nothing but Regge trajectories of t-(u-) channel exchanges that are proportional to corresponding Casimirs.

As an example we considered in detail the case of gluon-gluon scattering which was first treated by Kidonakis, Oderda and Sterman in [4]. Its colour structure is sufficiently complex as the problem involves in general six colour states (which reduce to five in SU(3)). We found a simple representation for arbitrary N for the eigenvalues of the matrix $\mathcal{Q}$, related with Γ as $\Gamma = -N(T+U)\,\mathcal{Q}$, with $T = \ln(s/|t|) - i\pi$, $U = \ln(s/|u|) - i\pi$. In our representation the three N-dependent energy levels (3.16) and corresponding eigenvectors (3.16) are explicitly real functions of T/U (the property not easy to extract from [4]).

We gave explicit solutions for the soft factor $\mathcal{S}_X$ in a number of special cases including large-N (3.40) and Regge limits (3.46).

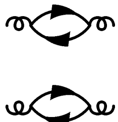
Finally, we observed that the cubic equation (3.13) for the N-dependent energy levels 4, 5, 6 of $\mathcal{Q}$ possesses a weird symmetry which interchanges internal (colour group) and external (scattering angle) degrees of freedom:

$$\frac{T+U}{T-U} \quad \Longleftrightarrow \quad N\,. \tag{4.3}$$

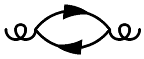
In particular, this symmetry relates 90-degree scattering, $T = U$, with the large-N limit of the theory. Giving the complexity of the expressions involved, such a symmetry being accidental looks highly improbable. Its origin remains mysterious and may point at existence of an enveloping theoretical context that correlates internal and external variables (string theory?).

A. Two gluon states in $SU(N)$

To construct colour states of the two-gluon system in the s-channel we draw a pictorial identity

$$\mathbb{1}^{bb'}_{aa'} \equiv \delta_{aa'}\,\delta^{bb'} =$$  $$= 4$$  (A.1)

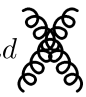
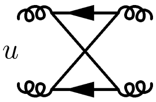
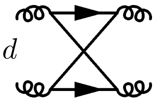
where a, b (a', b') are colour indices of incoming (outgoing) gluons, and analyse an intermediate state consisting of two quarks and two antiquarks. Here we have used

 $$= \quad \mathrm{tr}(t^a t^{a'}) = \frac{1}{2}\delta_{aa'}\,. \tag{A.2}$$

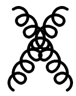
By interchanging quark and antiquark lines we can construct four tensors with a given symmetry with respect to quark and, separately, antiquark colour indices,

$$\mathbb{1} = \Pi^+_+ + \Pi^+_- + \Pi^-_+ + \Pi^-_-\,. \tag{A.3}$$

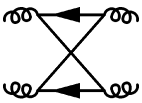
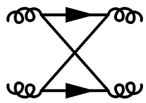
We get

$$\Pi^u_d = \frac{1}{4}\Big($$  $$+\,ud$$  $$\Big) + u$$  $$+\,d$$ 

where $u, d = \pm$ label the symmetry with respect to two "internal" quarks and two antiquarks, respectively. Introducing the notation

$$\mathbb{1} =$$  , $$\mathbf{X} =$$  $$\left(= \delta^b_{a'}\delta^{b'}_a \right),$$

and

$$\mathbf{W}_+ =$$  , $$\mathbf{W}_- =$$  $$\left(= \mathrm{Tr}(\, t^b\, t^{a'}\, t^a\, t^{b'}\,) \right),$$

we write down these four combinations as

$$\begin{aligned}
\Pi^+_+ &= \frac{1}{4}(\mathbb{1} + \mathbf{X}) + \mathbf{W}_+ + \mathbf{W}_- \\
\Pi^-_- &= \frac{1}{4}(\mathbb{1} + \mathbf{X}) - \mathbf{W}_+ - \mathbf{W}_- \\
\Pi^+_- &= \frac{1}{4}(\mathbb{1} - \mathbf{X}) + \mathbf{W}_+ - \mathbf{W}_- \\
\Pi^-_+ &= \frac{1}{4}(\mathbb{1} - \mathbf{X}) - \mathbf{W}_+ + \mathbf{W}_-\,.
\end{aligned} \tag{A.4}$$

For example, Π^+_+ is symmetric under interchanging quarks and antiquarks, Π^+_- is quark-symmetric and antisymmetric with respect to antiquarks, etc.

The sum of (A.4) obviously reproduces (A.3). Observing that interchanging the *gluons* $a \leftrightarrow b$ we have $\mathbb{1} \leftrightarrow \mathbf{X}$ and $\mathbf{W}_+ \leftrightarrow \mathbf{W}_-$, we conclude that

Π^+_+ and Π^-_- are symmetric,

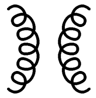
Π^+_- and Π^-_+ are antisymmetric

with respect to interchanging the gluon indices a, b (corresponding to $t \leftrightarrow u$).

A.1 Projectors

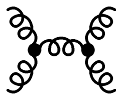
The simplest projectors — those onto the singlet and (antisymmetric an symmetric) octet states of two gluons — are obtained by connecting a quark and an antiquark line in (A.1) and can be represented graphically as

$$\mathcal{P}_{\mathbf{1}} = \frac{1}{N^2-1}$$



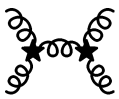
$$\left(= \frac{1}{N^2-1}\,\delta_a^b\,\delta_{a'}^{b'} \right), \tag{A.5}$$

$$\mathcal{P}_{\mathbf{a}} = \frac{1}{N}$$



$$\left(= \frac{1}{N}\,if_{abc}\,if_{cb'a'} \;=\; \frac{1}{N}\,T^c_{ab}T^c_{b'a'} \right), \tag{A.6}$$

$$\mathcal{P}_{\mathbf{s}} = \frac{N}{N^2-4}$$



$$\left(= \frac{N}{N^2-4}\,d_{abc}\,d_{a'b'c} \right), \tag{A.7}$$

where the dot in (A.6) marks the standard tree-gluon vertex (group structure constant) if_{abc} and the star in (A.7) stands for the symmetric d_{abc} symbol.

Introducing the notation for s-channel "multiplication" of two graphs A and B,

$$\left(A \cdot B\right)^{bb'}_{aa'} \;=\; \sum_{c,d} A^{bd}_{ac}\,B^{db'}_{ca'}\,,$$

one obtains [12]

$$\mathbf{W}_{\pm}\cdot\mathcal{P}_{\mathbf{1}} = -\frac{1}{4N}\mathcal{P}_{\mathbf{1}}\,, \qquad \mathbf{W}_{\pm}\cdot\mathcal{P}_{\mathbf{a}} = 0\,, \qquad \mathbf{W}_{\pm}\cdot\mathcal{P}_{\mathbf{s}} = -\frac{1}{2N}\mathcal{P}_{\mathbf{s}}\,. \tag{A.8}$$

These relations help us to construct remaining irreducible representations. To this end we subtract from the tensors Π^u_d (A.4) their projections onto the singlet and two octets, (A.5)–(A.7), and derive the four higher projectors:

$$\mathcal{P}_{27} = \Pi^+_+ - \frac{N-2}{2N}\mathcal{P}_{\mathbf{s}} - \frac{N-1}{2N}\mathcal{P}_{\mathbf{1}}\,; \tag{A.9}$$

$$\mathcal{P}_{0} = \Pi^-_- - \frac{N+2}{2N}\mathcal{P}_{\mathbf{s}} - \frac{N+1}{2N}\mathcal{P}_{\mathbf{1}}\,; \tag{A.10}$$

$$P_{10} = \Pi^+_- - \frac{1}{2}\mathcal{P}_{\mathbf{a}}\,, \qquad P_{\overline{10}} = \Pi^-_+ - \frac{1}{2}\mathcal{P}_{\mathbf{a}}\,. \tag{A.11}$$

For our purposes, the irreducible decuplet and anti-decuplet representations (A.11) can be handled as a single state, $\mathcal{P}_{\mathbf{10}} = P_{10} + P_{\overline{10}}$. Written in full, the higher representation projectors read

$$\mathcal{P}_{\mathbf{10}} = \frac{1}{2}(\mathbb{1} - \mathbf{X}) - \mathcal{P}_{\mathbf{a}}\,; \tag{A.12}$$

$$\mathcal{P}_{\mathbf{27}} = \frac{1}{4}(\mathbb{1} + \mathbf{X}) - \frac{N-2}{2N}\mathcal{P}_{\mathbf{s}} - \frac{N-1}{2N}\mathcal{P}_{\mathbf{1}} + (\mathbf{W}_+ + \mathbf{W}_-)\,; \tag{A.13}$$

$$\mathcal{P}_{\mathbf{0}} = \frac{1}{4}(\mathbb{1} + \mathbf{X}) - \frac{N+2}{2N}\mathcal{P}_{\mathbf{s}} - \frac{N+1}{2N}\mathcal{P}_{\mathbf{1}} - (\mathbf{W}_+ + \mathbf{W}_-)\,. \tag{A.14}$$

Making use of (A.8) and of the relations [12]

$$16\,\mathbf{W}_{\pm}\!\cdot\!\mathbf{W}_{\pm} = \mathbb{1} - \frac{N^2-1}{N^2}\mathcal{P}_{\mathbf{1}} - \frac{N^2-4}{N^2}\mathcal{P}_{\mathbf{s}} - \mathcal{P}_{\mathbf{a}}\,, \tag{A.15}$$

$$16\,\mathbf{W}_{\pm}\!\cdot\!\mathbf{W}_{\mp} = \mathbf{X} - \frac{N^2-1}{N^2}\mathcal{P}_{\mathbf{1}} - \frac{N^2-4}{N^2}\mathcal{P}_{\mathbf{s}} + \mathcal{P}_{\mathbf{a}}\,, \tag{A.16}$$

it is straightforward to verify that the operators (A.12)–(A.14) are indeed projectors,

$$\mathcal{P}_\alpha \cdot \mathcal{P}_\beta \;=\; \mathcal{P}_\alpha\,\delta_{\alpha\beta}\,. \tag{A.17}$$

A.2 Casimir operators

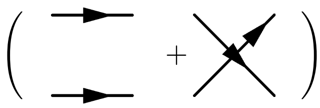
The Casimir operators for the singlet, $c_{\mathbf{1}} = 0$, and octet states, $c_{\mathbf{a}} = c_{\mathbf{s}} = N$, are known. To obtain c_R for higher representations $R = \mathbf{10}, \mathbf{27}, \mathbf{0}$ we construct the total colour charge of the two-gluon state as a sum of four quark generators,

$$T^a_R \;=\; t^a_1 + t^a_2 + \bar{t}^a_{1'} + \bar{t}^a_{2'}\,, \tag{A.18}$$

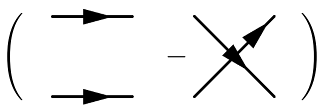
where $(1,2)$ and $(1',2')$ are "internal" colour lines of quarks and antiquarks inside two gluons, see (A.1). N^2 colour states of a qq pair (1,2) split into symmetric ($\mathbf{6}$) and antisymmetric ($\bar{\mathbf{3}}$) irreducible representations,

$$\equiv (3)\otimes(3) \;=\; P_{\mathbf{6}} \;+\; P_{\bar{\mathbf{3}}}\,; \tag{A.19a}$$

$$P_{\mathbf{6}} \;=\; \frac{1}{2}\Big(\quad + \quad\Big) \tag{A.19b}$$



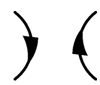
$$P_{\bar{\mathbf{3}}} \;=\; \frac{1}{2}\Big(\quad - \quad\Big)\,. \tag{A.19c}$$



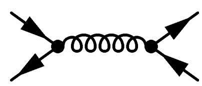
A $q\bar{q}$ pair $((1,1'),(1,2'),(2,1'),(2,2'))$ can be in general in the colour singlet ($\mathbf{1}$) and colour octet ($\mathbf{8}$) characterised by the projectors

$$\equiv (3)\otimes(\bar{3}) \;=\; P_{\mathbf{1}} \;+\; P_{\mathbf{8}}\,; \tag{A.20a}$$

$$P_{\mathbf{1}} \;=\; \frac{1}{N} \tag{A.20b}$$



$$P_{\mathbf{8}} \;=\; 2 \tag{A.20c}$$



Remark that (A.20) is nothing but the graphic representation of the Fierz identity.

Squaring T^a_R in (A.18) and exploiting the symmetry properties of the representations (A.9)–(A.11) with respect to quarks and antiquarks we arrive at

$$c_R \equiv (T^a_R)^2 \;=\; 4C_F + 8v_{\mathbf{3}\bar{\mathbf{3}}}(\mathbf{8}) + 2\,[\,v_{\mathbf{33}}(R_q) + v_{\bar{\mathbf{3}}\bar{\mathbf{3}}}(R_{\bar{q}})\,]\,, \tag{A.21}$$

where v is the one-gluon "exchange potential" between fermions, and R_q ($R_{\bar{q}}$) marks the irreducible SU(3) representation of the two-quark (antiquark) system.

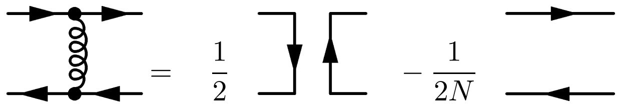
Since our representations are *traceless*, an internal quark and an antiquark are always in the *octet* state,

$$v_{\mathbf{3\bar{3}}}P_{\mathbf{8}} = v_{\mathbf{3\bar{3}}}(\mathbf{8})P_{\mathbf{8}} = t^a\,\bar{t}^a \cdot P_{\mathbf{8}} = -t^a\,t^a \cdot P_{\mathbf{8}} = \frac{1}{2N}\,\mathcal{P}_{\mathbf{8}}\,.$$

R_q may be either $\mathbf{6}$ or $\bar{\mathbf{3}}$ depending on the quark symmetry ($\bar{\mathbf{6}}$ and $\mathbf{3}$ for $R_{\bar{q}}$):

$$v_{\mathbf{33}}(\mathbf{6}) = v_{\mathbf{\bar{3}\bar{3}}}(\bar{\mathbf{6}}) = -\frac{1-N}{2N}\,, \qquad v_{\mathbf{33}}(\bar{\mathbf{3}}) = v_{\mathbf{\bar{3}\bar{3}}}(\mathbf{3}) = -\frac{1+N}{2N}\,. \tag{A.22}$$

The values of inter-quark potentials $v_{33}(\mathbf{6})$ and $v_{33}(\bar{\mathbf{3}})$, and $v_{3\bar{3}}(\mathbf{8})$ between a quark and an antiquark are easy to derive by projecting the Fierz identity (A.20) in the "rotated" form,



, (A.23)

onto the s-channel states (A.19) and (A.20c). Thus we obtain

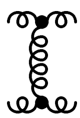
$$c_R \;=\; 4\frac{N^2-1}{2N} + \frac{4}{N} + 2\left(\frac{u\,N-1}{2N} + \frac{d\,N-1}{2N}\right), \tag{A.24}$$

giving

$$\begin{aligned} c_{\mathbf{27}} &= 2(N+1)\,, && (u = d = +) \\ c_{\mathbf{0}} &= 2(N-1)\,, && (u = d = -) \\ c_{\mathbf{10}} &= 2N\,. && (u = -d)\,. \end{aligned} \tag{A.25}$$

A.3 t- and u-channel projectors in terms of s-channel ones

t-channel gluon exchange between gluons (s-channel gluon exchange potential) is proportional to the t-channel antisymmetric octet projector $\mathcal{P}_{\mathbf{a}}^{(t)}$ and has the following decomposition in s-channel colour projectors

$V \equiv$ 

$$= N\mathcal{P}_{\mathbf{a}}^{(t)} = \frac{N}{2}\cdot\mathcal{P}_{\mathbf{a}} + 0\cdot\mathcal{P}_{\mathbf{10}} + N\cdot\mathcal{P}_{\mathbf{1}} + \frac{N}{2}\cdot\mathcal{P}_{\mathbf{s}} + (-1)\cdot\mathcal{P}_{\mathbf{27}} + 1\cdot\mathcal{P}_{\mathbf{0}}. \tag{A.26}$$

The coefficients in (A.26) are related with the Casimir operators of the representations involved, $c_\alpha = (T_1+T_2)^2$ and $T_1^2 = T_2^2 = N$, as

$$V\cdot\mathcal{P}_\alpha \;=\; \frac{1}{2}\,[\,2N - c_\alpha\,]\cdot\mathcal{P}_\alpha\,, \tag{A.27}$$

with c_α given in (3.7). The u-channel projector $\mathcal{P}_{\mathbf{a}}^{(u)}$ is obtained from (A.26) by transposing the outgoing gluons which amount to changing sign of the asymmetric components $\mathcal{P}_{\mathbf{a}}$ and $\mathcal{P}_{\mathbf{10}}$.

The re-projection matrix K_{ts} introduced in (2.31a) is given by

$$K_{ts} = \begin{pmatrix} \frac{1}{2} & 0 & 1 & \frac{1}{2} & -\frac{1}{N} & \frac{1}{N} \\ 0 & \frac{1}{2} & \frac{N^2-4}{2} & -1 & -\frac{N-2}{2N} & -\frac{N+2}{2N} \\ \frac{1}{N^2-1} & \frac{1}{N^2-1} & \frac{1}{N^2-1} & \frac{1}{N^2-1} & \frac{1}{N^2-1} & \frac{1}{N^2-1} \\ \frac{1}{2} & -\frac{2}{N^2-4} & 1 & \frac{N^2-12}{2(N^2-4)} & \frac{1}{N+2} & -\frac{1}{N-2} \\ -\frac{N(N+3)}{4(N+1)} & -\frac{N(N+3)}{4(N+1)(N+2)} & \frac{N^2(N+3)}{4(N+1)} & \frac{N^2(N+3)}{4(N+1)(N+2)} & \frac{N^2+N+2}{4(N+1)(N+2)} & \frac{N+3}{4(N+1)} \\ \frac{N(N-3)}{4(N-1)} & -\frac{N(N-3)}{4(N-1)(N-2)} & \frac{N^2(N-3)}{4(N-1)} & -\frac{N^2(N-3)}{4(N-1)(N-2)} & \frac{N-3}{4(N-1)} & \frac{N^2-N+2}{4(N-1)(N-2)} \end{pmatrix} \qquad \text{(A.28)}$$

After some reflection it is easy to understand why does the inverse matrix coincide with the direct one:

$$K_{st} \;=\; K_{ts}\,. \qquad \text{(A.29)}$$

It is also easy to construct the u-channel re-projection matrices (2.31b) exploiting the symmetry of the s-channel projectors under $t \leftrightarrow u$ transformation. Thus K_{us} is obtained by changing sign of the first two *columns* of K_{ts} and the inverse matrix K_{su} — by changing sign of the first two *rows* of K_{ts}.

Hints for deriving (A.28) **without spilling much blood [12].**

$\mathcal{P}^{(t)}_{\mathbf{a}}$: is $1/N$ times the s-channel "gluon exchange potential"; see (A.26).

$\mathcal{P}^{(t)}_{\mathbf{10}}$: look at (A.12) from t-channel perspective,

$$\mathcal{P}^{(t)}_{\mathbf{10}} = \frac{1}{2}(\mathbb{1}^{(t)} - \mathbf{X}^{(t)}) - \mathcal{P}^{(t)}_{\mathbf{a}}\,; \qquad \mathbb{1}^{(t)} \equiv (N^2-1)\cdot\mathcal{P}_{\mathbf{1}}\,, \qquad \mathbf{X}^{(t)} \equiv \mathbf{X}\,.$$

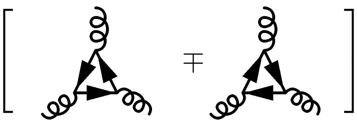
Representation of the last term $\mathcal{P}^{(t)}_{\mathbf{a}}$ we already know. The cross we get from the representation of the unity, by exchanging $t \leftrightarrow u$:

$$\begin{aligned} \mathbb{1} &= \;\;\mathcal{P}_{\mathbf{a}} + \mathcal{P}_{\mathbf{10}} + \mathcal{P}_{\mathbf{1}} + \mathcal{P}_{\mathbf{s}} + \mathcal{P}_{\mathbf{27}} + \mathcal{P}_{\mathbf{0}}, \\ \mathbf{X} &= -\mathcal{P}_{\mathbf{a}} - \mathcal{P}_{\mathbf{10}} + \mathcal{P}_{\mathbf{1}} + \mathcal{P}_{\mathbf{s}} + \mathcal{P}_{\mathbf{27}} + \mathcal{P}_{\mathbf{0}}. \end{aligned}$$

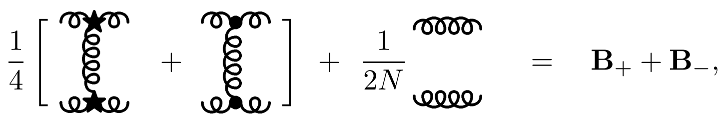
$\mathcal{P}^{(t)}_{\mathbf{1}}$: $\equiv (N^2-1)^{-1}\cdot\mathbb{1}$.

$\mathcal{P}^{(t)}_{\mathbf{s}}$: using pictorial representation for if_{abc} and d_{abc} symbols in terms of quark loops,

$$if_{abc}\,[d_{abc}] = 2\,[\;\ldots\;\mp\;\ldots\;]\,, \qquad \text{(A.30)}$$



and the Fierz identity (A.23) to get rid of gluons connecting quark triangles, it is straightforward to derive



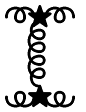
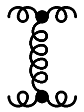
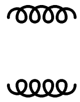
$$\ldots \;=\; \mathbf{B}_+ + \mathbf{B}_-, \qquad \text{(A.31)}$$

where $\mathbf{B}_\pm$ stand for *quark boxes* with positive (negative) direction of the fermion line: $\mathbf{B}_+ = \mathrm{Tr}[t^{a_2}t^{a_1}t^{a_3}t^{a_4}]$, $\mathbf{B}_- = \mathrm{Tr}[t^{a_1}t^{a_2}t^{a_4}t^{a_3}]$. Since the boxes are rotationally invariant, we can *rotate* the l.h.s. by 90° and write the same expression in terms of the s-channel operators:

$$\mathbf{B}_+ + \mathbf{B}_- \;\equiv\; \mathbf{B}_+^{(t)} + \mathbf{B}_-^{(t)} \;=\; \frac{N}{4}\mathcal{P}_{\mathbf{a}} + \frac{N^2-1}{2N}\mathcal{P}_{\mathbf{1}} + \frac{N^2-4}{4N}\mathcal{P}_{\mathbf{s}}\,. \tag{A.32}$$

Equating (A.31)=(A.32), and already knowing $\mathcal{P}_{\mathbf{a}}^{(t)}$, suffices to get hold of $\mathcal{P}_{\mathbf{s}}^{(t)}$.

$\mathcal{P}_{\mathbf{27},\mathbf{0}}^{(t)}$: for the t-channel **27** and **0** projectors we use "rotated" definitions (A.13) and (A.14). In these expressions we know all the elements but the sum of rotated crossed boxes $W_+^{(t)} + W_-^{(t)}$. Flipping (A.31) around the top (that is, exchanging $1 \leftrightarrow 3$) we have

$\frac{1}{4}\Big[$  $-$  $\Big] + \frac{1}{2N}$  $= \mathbf{W}_+ + \mathbf{W}_-\,.$ (A.33)

"Rotating" (A.33) by 90° we obtain

$$\mathbf{W}_+^{(t)} + \mathbf{W}_-^{(t)} \;=\; -\frac{N}{4}\mathcal{P}_{\mathbf{a}} + \frac{N^2-1}{2N}\mathcal{P}_{\mathbf{1}} + \frac{N^2-4}{4N}\mathcal{P}_{\mathbf{s}}\,. \tag{A.34}$$

Finally, it does not hurt to check the t-channel completeness relation: $\sum_\rho \mathcal{P}_\rho^{(t)} = \mathbb{1}^{(t)}$.

B. Symmetric basis

The soft anomalous dimension matrix $\mathcal{Q}$ can be made symmetric by applying the metric operation which consists of multiplying the columns and dividing the rows by the square root of the dimension of the representation. That is,

$$\Gamma^{\mathrm{symm}} = \mathcal{M}^{-1}\,\Gamma\,\mathcal{M}\,, \qquad \mathcal{M}_{\alpha\beta} = \sqrt{K_\alpha}\,\delta_{\alpha\beta}\,,$$

with K_α given in (3.3). In the symmetrised form the $\mathcal{Q}$ matrix reads

$$\mathcal{Q}^{(s)} = \left(\begin{array}{cc|cccc}
\frac{3}{2} & 0 & -\frac{2b}{U_1D_1} & -\frac{b}{2} & -\frac{b\,U_3}{N\,U_1} & -\frac{b\,D_3}{N\,D_1} \\
0 & 1 & 0 & -\frac{b\sqrt{2}}{U_2D_2} & -\frac{b\,U_1D_2U_3}{\sqrt{2}N\,U_2} & -\frac{b\,D_1U_2D_3}{\sqrt{2}N\,D_2} \\
\hline
-\frac{2b}{U_1D_1} & 0 & 2 & 0 & 0 & 0 \\
-\frac{b}{2} & -\frac{b\sqrt{2}}{U_2D_2} & 0 & \frac{3}{2} & 0 & 0 \\
-\frac{b\,U_3}{N\,U_1} & -\frac{b\,U_1D_2U_3}{\sqrt{2}N\,U_2} & 0 & 0 & \frac{N-1}{N} & 0 \\
-\frac{b\,D_3}{N\,D_1} & -\frac{b\,D_1U_2D_3}{\sqrt{2}N\,D_2} & 0 & 0 & 0 & \frac{N+1}{N}
\end{array}\right), \tag{B.1}$$

where we used shorthand notation

$$U_k = \sqrt{N+k}\,, \quad D_k = \sqrt{N-k}\,.$$

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